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Analytic signals and the transition to chaos in deterministic flows

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The transition from regular to chaotic motions in deterministic flows is characterized by a change from a discrete Fourier spectrum to a broadband one. The onset of chaos is thus associated with the creation of an infinite number of new Fourier modes. Given a system that generates a time series $x(t)$, we study the transition to chaos from the perspective of analytic signals, which are defined via the Hilbert transform. In order to identify distinct analytic signals, we decompose the original time series $x(t)$ into a finite number of modes that correspond to proper rotations in the complex plane of their analytic signals. We provide numerical evidence that at the transition, there is no substantial change in the number of analytic signals characterizing $x(t)$. Furthermore, the distributions of the instantaneous frequencies of the analytic signals in the chaotic regime are well localized and exhibit no broadband feature. These results suggest a simple organization of chaos in terms of analytic signals. [S1063-651X(98)50712-X]

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Turbulent and chaotic motions occur commonly in many natural processes. A fundamental question then concerns how these motions occur as a system parameter changes. About a half century ago, Landau proposed that turbulent motion was a result of the successive addition of a great many new discrete frequency components as the system parameter approaches the critical point [1] at the onset of the turbulent or chaotic motion [2]. This turbulence scenario, however, was shown to be incorrect by Ruelle, Takens, and Newhouse [3] who proved mathematical theorems concerning the transition to chaotic motion from four- and three-frequency quasiperiodic flows. The key implication of their results is that broadband frequency spectra, a hallmark of turbulent and chaotic motions, can appear more abruptly as the result of the onset of a chaotic attractor. To be more specific, consider a physical system described by a continuous flow:

$$\frac{d\mathbf{x}}{dt}=\mathbf{F}(\mathbf{x},p), \tag{1}$$

where $\mathbf{x}\in\mathbf{R}^N$ and p is a system parameter. Assume at $p=p_1$ that the flow is quasiperiodic. In this case, if one examines the Fourier spectrum of $\mathbf{x}(t)$, one finds only a few incommensurate Fourier frequencies. According to Ruelle, Takens, and Newhouse [3], an arbitrarily small change, say in the parameter p from p_1 to $p_2=p_1+\delta p$, where $\delta p\sim 0$, can lead to a chaotic motion characterized by a broadband Fourier spectrum. Note that there are in fact an infinite number of incommensurate Fourier frequencies associated with the chaotic motion at p_2, whereas there are only a very few such frequencies at p_1 even if $|p_2-p_1|\to 0$. Thus, an infinite number of fundamental Fourier frequencies must have been created through an arbitrarily small parameter change.

In this paper, we address the transition to chaos in deterministic flows from the perspective of analytic signals, a concept that was originally proposed by Gabor in optics [4]. Given a scalar time series $x(t)$, obtained from a measurement of a nonlinear system, the corresponding analytic signal is defined to be

$$\psi_x(t)=x(t)+iH[x(t)], \tag{2}$$

where the imaginary part $H[x(t)]$ is the Hilbert transform of $x(t)$:

$$H[x(t)]=\mathcal{P}\left[\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{x(t')}{t-t'}\,dt'\right]. \tag{3}$$

In Eq. (3), the notion $\mathcal{P}$ stands for the Cauchy principal value of the integral. The analytic signal represents a rotation in the complex plane $\{x(t),H[x(t)]\}$, with instantaneous frequency given by $\omega(t)=d\phi(t)/dt$, where the phase angle $\phi(t)$ is defined through the representation $\psi_x(t)=A(t)\exp[i\phi(t)]$, and $A(t)$ is the instantaneous radius of the rotation. The instantaneous frequency so defined, however, may possess negative values [5]. In order to only have positive frequencies that are physically meaningful, and to detect distinct fundamental frequency components, it is necessary to preprocess the time series $x(t)$. We use the *empirical-mode* decomposition method developed by Huang *et al.* [5] to decompose $x(t)$ into a finite number of components whose analytic signals yield only positive instantaneous frequencies. The principal results of this paper are: (1) distributions of the instantaneous frequencies for a chaotic system are typically well localized and exhibit no broadband feature, in contrast to the Fourier spectra of chaotic signals, and (2) there is no substantial change in the number of analytic signals that constitute a dynamical variable before and after the onset of chaos. The implication is that transition to chaos in nonlinear systems can be considered as a rather smooth process when the transition is viewed from the perspective of analytic signals rather than from that of the traditional Fourier spectra. Our results also suggest an interesting organization of chaos in continuous flows, that is, chaos is supported by only a few distinct rotations in the complex representations of analytic signals.

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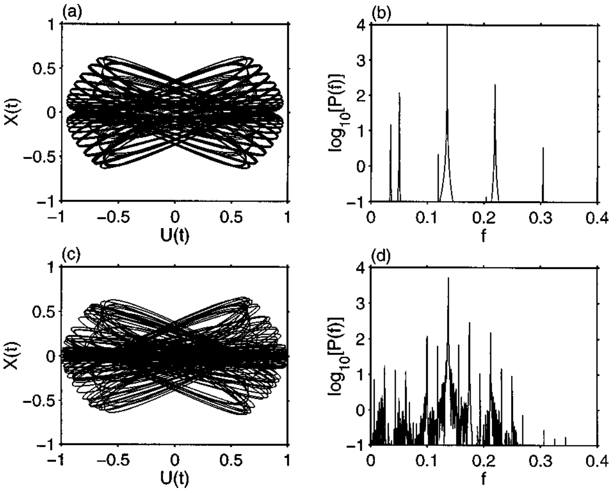

FIG. 1. (a) For $p=0.25$ (two-frequency quasiperiodicity), a trajectory in the (U,X) plane. (b) The Fourier spectrum of $U(t)$ in (a). (c) For $p=0.24$ (chaos), a trajectory in the (U,X) plane. (d) The Fourier spectrum of $U(t)$ in (c).

We have studied the analytic-signal representation of dynamic variables for several chaotic systems including the Lorenz system [6] and the Rössler system [7]. Here we choose to present our results with the model of two-mode truncation of the complex coefficient Ginzburg-Landau equation [8]. The model represents a four-dimensional autonomous flow:

$$\frac{da_1}{dt}=pa_1+(i-p)(|a_1|^2a_1+a_1|a_2|^2+\tfrac{1}{2}a_1^*a_2^2),$$
$$\frac{da_2}{dt}=pa_2-q^2(i+p)a_2+(i-p)(a_1^2a_2^*+2|a_1|^2a_2+\tfrac{3}{4}|a_2|^2a_2), \quad (4)$$

where $a_1(t)$ and $a_2(t)$ are complex dynamical variables, the star denotes a complex conjugate, and p and q are parameters. It was argued that Eq. (4) exhibits a transition from two-frequency quasiperiodic motion to chaos in wide parameter regimes via the mechanism of heteroclinic crossing of stable and unstable manifolds and torus breakup [8]. In particular, it was demonstrated that a two-frequency quasiperiodic motion can lose its stability directly and becomes chaotic. In the sequel we fix $q=1.0$ and use the following notion: $U(t)\equiv\mathrm{Re}[a_1(t)]$, $V(t)\equiv\mathrm{Im}[a_1(t)]$, $X(t)\equiv\mathrm{Re}[a_2(t)]$, and $Y(t)\equiv\mathrm{Im}[a_2(t)]$. Numerical computation indicates that the transition from two-frequency quasiperiodicity to chaos occurs at the critical parameter value $0.24<p_c<0.25$, where the motion is quasiperiodic for $p>p_c$ and chaotic for $p<p_c$. Figure 1(a) shows, for $p=0.25$, the projection of the quasiperiodic attractor (after a transient time of $t=50\,000$) onto the (U,X) plane, and Fig. 1(b) shows the Fourier power spectrum of $U(t)$ for $0\leqslant t\leqslant 3276.8$ at a sampling rate $\Delta t=0.05$ (so that there are 2^{16} points in the time series for fast Fourier transform). Numerical integration of Eq. (4) was carried out by using a fifth-order adaptive stepsize Runge-Kutta algorithm. We see that the Fourier spectrum is apparently discrete. Examination of the spectrum indicates that there are two fundamental frequencies [8]. As p decreases, the two-frequency torus in which the quasiperiodic attractor lies breaks at p_c, and the asymptotic attractor becomes chaotic with a fractal dimension between 3 and 4 for $p<p_c$ [8]. Figures 1(c) and 1(d) show the projection of the chaotic attractor in the (U,X) plane and the Fourier power spectrum of $U(t)$, respectively. Clearly, the Fourier spectrum now has a broadband feature, which is a hallmark of chaos. Comparison between Figs. 1(b) and 1(d) indicates that an infinite number of new Fourier modes is created at the onset of chaos.

We now examine the transition to chaos, as demonstrated in Figs. 1(a)–1(d), from the standpoint of analytic signals. In order to obtain analytic signals with positive instantaneous frequencies from a time series (or signal) $x(t)$, it is necessary to decompose the signal in a proper way. To gain intuition, imagine a counterclockwise (or clockwise) rotation of a particle on a circle of unit radius in the plane. This motion can be characterized by an angle function, or phase, $\phi(t)=\omega(t)t$, where $\omega(t)$ is the instantaneous frequency of the rotation that satisfies $\omega(t)\geqslant 0$. The position of the particle on a line passing through the center of the circle can be described by the function $\cos[\phi(t)]$ or $\sin[\phi(t)]$. For such functions that describe rotation, the number of maxima and minima is equal to the number of zeros in a given large time interval, a property that defines a *proper rotation.* This observation provides a general principle to obtain components with proper analytic signals from a complicated signal $x(t)$. The empirical-mode-decomposition method [5] we employ consists of three steps: (1) construct two smooth splines connecting all the maxima and minima of $x(t)$ to get $x_{\max}(t)$ and $x_{\min}(t)$, respectively; (2) compute $\Delta x(t)\equiv x(t)-[x_{\max}(t)+x_{\min}(t)]/2$; and (3) repeat steps (1) and (2) for $\Delta x(t)$ until the resulting signal corresponds to a proper rotation. Denote the resulting signal by $C_1(t)$, which is the first component of $x(t)$. We then take the difference $x_1(t)\equiv x(t)-C_1(t)$ and repeat steps (1)–(3) to obtain the second component $C_2(t)$ from $x_1(t)$. The procedure continues until the component $C_M(t)$ shows no apparent time variation [5]. The original signal $x(t)$ can thus be expressed as $x(t)=\Sigma_{j=1}^{M}C_j(t)$, where the functions $C_j(t)$ are nearly orthogonal to each other [5]. By the nature of the decomposition procedure, the first component $C_1(t)$ corresponds to the fastest time variation of $x(t)$ and, hence, the signal has the smallest time scale. As the mode index j increases, the time scale increases so that the mean frequency of the rotation decreases. While this procedure is generally applicable to a smooth signal $x(t)$, we stress that it is only *empirical* [5]. There are also situations in which the variable $x(t)$ itself is already a proper rotation. In such a case, the procedure decomposes $x(t)$ into components with distinct time scales, and the analytic signals of which yield distinct mean frequencies.

After $x(t)$ is decomposed, one can obtain quantitative characteristics of the rotations from the analytic signal of each component by utilizing the Hilbert transform. To gain insight about the nature of the Hilbert transform, consider the mathematical function $e^{i\omega t}$, which is the simplest analytic signal. If one plots the real and the imaginary parts of $e^{i\omega t}$ in the complex plane of $e^{i\omega t}$, one obtains a rotation with angular frequency ω. This example also illustrates the role of the Hilbert transform: for a simple harmonic function $\cos(\omega t)$, the Hilbert transform simply shifts the phase of the function

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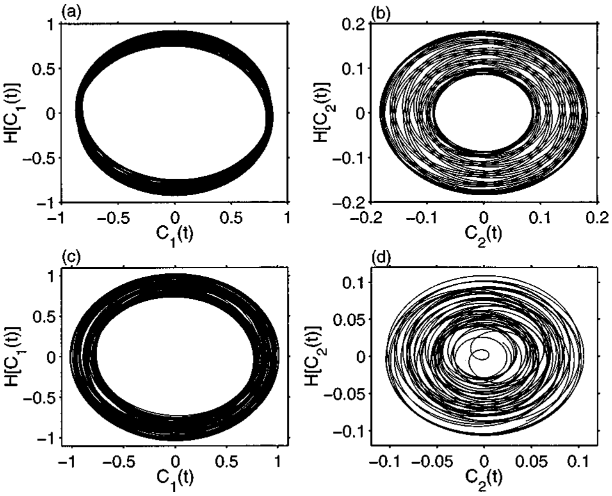

FIG. 2. The first two proper analytic signals obtained from $U(t)$. (a) $p=0.25$ (quasiperiodicity), mode 1; (b) $p=0.25$ (quasiperiodicity), mode 2; (c) $p=0.24$ (chaos), mode 1; and (d) $p=0.24$ (chaos), mode 2.

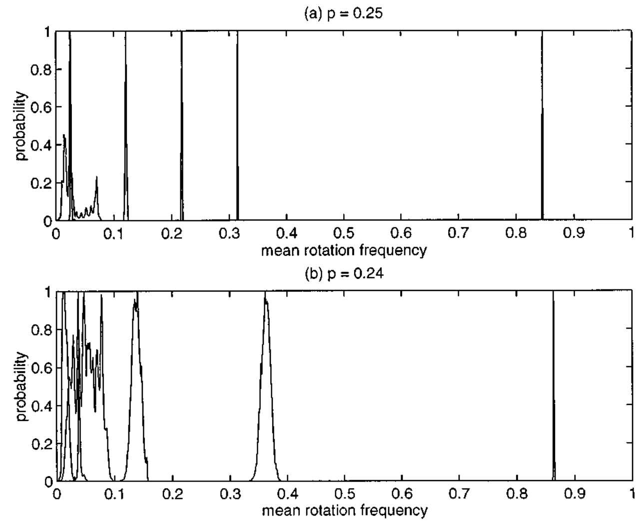

FIG. 3. Histograms of mean frequencies of proper analytic signals associated with the $U(t)$ for (a) the quasiperiodic motion at $p=0.25$ and (b) for the chaotic motion at $p=0.24$.

by $\pi/2$. For a more complicated signal $C(t)$, one can numerically obtain its Hilbert transform via the following three steps: (1) decompose $C(t)$ into a large number of harmonics using the Fourier transform; (2) shift the phase of each harmonic component by $\pi/2$; and (3) sum up all of the phase-shifted harmonics.

We have analyzed trajectories for both the quasiperiodic and chaotic motions in Eq. (4) by decomposing the time series into components with proper analytic signals. Here we present results with the time series of one of the dynamical variables, say, $U(t)$. Specifically, we write $U(t)=\Sigma_{i=1}^{M}C_i(t)$, where M is the number of modes with nonzero mean frequencies of rotation. For a time series of 2^{16} points at a sampling rate of $\Delta t=0.05$, we find that $M\approx 6$ suffices to capture the time variation of the original signal $U(t)$. Figures 2(a) and 2(b) show, for $p=0.25$, the first two rotations in the complex planes of their own analytic signals. The average frequencies of these two rotations are $\omega_1\approx 0.846$ and $\omega_2\approx 0.314$, respectively. The rotations reveal rather regular patterns, as can be expected for a quasiperiodic motion. As p decreases passing through p_c so that the system is in a chaotic regime, these proper analytic signals still persist. Figures 2(c) and 2(d) show the corresponding rotations for $p=0.24$. Due to chaos, the rotations no longer exhibit regular patterns, but the overall behaviors of rotation still exist. The average frequencies of rotation are $\omega_1\approx 0.864$ and $\omega_2\approx 0.378$ for modes 1 and 2, respectively. We see that the mean frequency of the first analytic signal changes only slightly as p decreases from 0.25 to 0.24, and the corresponding change in the second analytic signal is rather large. In general, we observe that the onset of chaos usually has a greater influence on rotations with smaller frequencies.

The remarkable result is that as the quasiperiodic motion is converted into a chaotic one, the number of proper analytic signals characterizing a chaotic signal essentially remains the same. For instance, we find that the chaotic signal $U(t)$ in Fig. 1(c) can still be represented by six proper analytic signals. To better examine the change in proper analytic signals for quasiperiodic and chaotic signals, we study the statistical behavior of mean frequencies of rotation. Specifically, for a fixed parameter value p, we choose 10 000 trajectories (each of 2^{16} points at a sampling rate $\Delta t=0.05$) on the attractor and for each trajectory we compute mean frequencies of the first six modes of proper rotations. The histograms of these six frequencies $(\omega_1,\ldots,\omega_6)$ are then computed. Figures 3(a)and 3(b) show the histograms for $p=0.25$ and $p=0.24$, respectively. When the motion is quasiperiodic [Fig. 3(a)], we see that the first few frequencies are sharply distributed. Note that some of the frequencies are the linear combinations of others, which occur commonly in the Fourier analysis. For instance, we find $\omega_3\approx(\omega_2+\omega_4)/2$. For the chaotic motion [Fig. 3(b)], the frequencies *spread* and the frequency distributions shift relative to those in the quasiperiodic case. However, the first few frequency distributions in the chaotic case are still well localized. One important feature distinguishing a chaotic rotation from a regular one is that for a chaotic rotation $\psi(t)=A(t)\exp[i\phi(t)]$, the amplitude $A(t)$ is random and the phase dynamics are similar to a random walk. This is due to the fact that the phase dynamics can be described by $d\phi(t)/dt=\omega+F[A(t)]$, where $F[A(t)]$ is a function of $A(t)$ [9,10]. The most important feature of Figs. 3(a) and 3(b) is that the number of proper analytic signals remains essentially unchanged through the transition from quasiperiodicity to chaos, and the distributions of the instantaneous frequencies of the analytic signals are *well localized* and exhibit no broadband feature. We speculate that a reason may be that in the rotation representation, the chaotic amplitude modulation is filtered out so that the broadband component in the Fourier spectrum disappears [11]. Thus, although an infinite number of Fourier modes is created at the onset of chaos, there is no metamorphosis in the number of analytic signals that represent a chaotic time series [12].

We remark that the frequencies ω_j obtained here correspond to the mean rotation frequencies of the empirical models $C_j(t)$ $(j=1,\ldots,M)$ in the complex planes of their analytic signals. These frequencies, in fact, characterize the main physical time scales hidden in the original time series $U(t)$ *from the perspective of rotations.* The frequencies ω_j can be rationally related and we observe that there are only two incommensurate ones, which is consistent with the fact that the underlying flow is two-frequency quasiperiodic. The Fourier frequencies, on the other hand, are harmonic frequencies. When the motion is regular, we expect the fundamental frequencies in the analytic-signal representation to be approximately equal to these in the Fourier representation, or the ''real'' frequencies of the quasiperiodic motion [compare Fig. 1(b) with Fig. 3(a)]. When the motion is chaotic, the Fourier spectrum is broadband, but the frequency distribution in the analytic-signal representation is still well localized, due to the fact that there cannot be abrupt change in the analytic signals (rotations) that constitute the physical signal. In this case, there is no direct correspondence between the rotation frequencies of the analytic signals and the Fourier frequencies.

In summary, we have examined the transition to chaos in deterministic flows from the standpoint of analytic signals. By studying the rotational characteristics of the analytic signals, we find that there is no significant change in the number of proper analytic signals through the transition, although the Fourier spectrum becomes broadband after the onset of chaos. Thus, although chaotic motion can be characterized as random and complicated, its fundamental structure in terms of proper analytic signals can be quite simple.

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[2] Here we use the word *turbulent* to mean random and complicated motions, such as a chaotic motion, as used by Landau [1].

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[11] By filtering out the chaotic amplitude modulations, the organization of the phase dynamics, which is characterized by the rotation frequencies, becomes quite simple in the sense that the distributions of frequencies are well isolated [Fig. 3(b)], even in chaotic regimes. A somewhat analogous situation occurs in the periodic-orbit representation of chaotic sets. It is believed that the infinite set of unstable periodic orbits constitutes the skeleton of a chaotic set [see, for example, Chaos **2**, 1 (1992)]. In this sense, chaos can be regarded as being organized on the infinite set of periodic orbits, but apparently there is an even larger set of *aperiodic* orbits embedded in the set. In our case, by focusing on the rotational characteristics of a chaotic flow, its organization becomes simple.

[12] This conclusion applies to chaotic motions in continuous flows only, as the notion of rotation is meaningful for flows, but not for discrete maps.